Replica symmetry breaking in and around six dimensions

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Abstract

Two, replica symmetry breaking specific, quantities of the Ising spin glass — the breakpoint x_1 of the order parameter function and the Almeida–Thouless line — are calculated in six dimensions (the upper critical dimension of the replicated field theory used), and also below and above it. The results confirm that replica symmetry breaking does exist below $d = 6$, and also the tendency of its escalation for decreasing dimension continues. As a new feature, x_1 has a nonzero and universal value for $d < 6$ at criticality. Near six dimensions we have $x_{1c} = 3(6 - d) + O[(6 - d)^2]$. A method to expand a generic theory with replica equivalence around the replica symmetric one is also demonstrated.
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1. Introduction

Frustration in disordered systems gives rise to a complex equilibrium state with a nontrivial breaking of ergodicity (see [1] for a review and important reprints of the field). In the mean field version of the Ising spin glass [2], the decomposition of the Gibbs state into ultrametrically organized pure states is (mathematically) encoded in the replica symmetry broken (RSB) solution of the replicated system [1]. This solution has characteristics — such as the order parameter *function* $q(x)$, and the spin glass transition in nonzero external magnetic field along the so-called

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Almeida–Thouless (AT) line — which fully distinguish it from the much simpler replica symmetric (RS) case. This RS solution is unstable in the mean field glassy phase [3].

From the physical point of view, RSB implies the presence of violations of nontrivial fluctuation–dissipation relations at off-equilibrium (during aging), while the off-equilibrium fluctuation–dissipation relations would be trivial in the RS case: in particular no aging of the response function is expected then, in variance with the experimental evidence in three dimensions at zero magnetic field. It is a very important task to determine the dimensional regime where the low temperature phase with aging response function survives. Evidently, there is no glassy phase in the one-dimensional system, whereas there is an ample numerical evidence against any transition in the two-dimensional case too. Generally speaking, we expect that the transitions disappear at the corresponding lower critical dimensions, i.e at d_{SG}^0 in zero magnetic field, and at d_{SG}^h in the presence of a magnetic field. We cannot say a priori if these two lower critical dimensions are the same: in the case of an Ising ferromagnet with a random magnetic field, for instance, it is well known that $d_{IF}^0 = 1$, whereas $d_{IF}^h = 0$. The situation in spin glasses is quite unclear: the different structure of the low momentum singularities in zero and nonzero magnetic field [4] suggest that $d_{SG}^0 < d_{SG}^h$, while the arguments based on domain wall energies give $d_{SG}^0 = d_{SG}^h = 2.5$ [5]. The existence of a low temperature phase with aging response function should be ultimately decided by investigating the structure of infrared divergences in the perturbative expansion, and by the analysis of nonperturbative contributions. This task goes by far beyond the goals of the present paper. We aim to study in details the properties of the low temperature phase near the critical temperature, and around the upper critical dimension (i.e. six) where the critical exponents at zero magnetic field become nontrivial. Our study also aims to correct some recent claims on the nonexistence of a RSB phase below six dimensions that are due to an incorrect analysis of the consequences of some renormalization group equations [6].

The mean field Ising spin glass, at least when studied with the replica trick, can be considered as the infinite-dimensional limit of the replica field theory representing the d-dimensional short ranged model defined on a hypercubic lattice [7]. The study of this replica field theory for decreasing dimensionalities seems to be a good strategy for reaching a full understanding of the three-dimensional Ising spin glass.

This project has had by now a long history whose first period was summarized in Ref. [8]. It turns out from these studies that the RS glassy phase is notoriously unstable even down to $d \lesssim 6$, with a persistently escalating RSB phase (see, for instance, Fig. 1 of Ref. [9]). A scaling picture was proposed in [10] for helping to understand one-loop calculations in the (zero external magnetic field) RSB phase. Some of the results of this reference are reproduced and/or revised in the present paper, especially the behaviour of the breakpoint x_1 of $q(x)$ around six dimensions. The AT line was first found in Ref. [11] for the range $6 < d < 8$, whereas it was followed up from mean field ($d = \infty$) to $d \lesssim 6$ (and also for nonzero replica number n) in [7].

Nevertheless, the RS spin glass phase has remained an alternative due to the so-called droplet model [12–14]. This theory predicts a unique Gibbs state (apart from spin inversion) for $T < T_c$ — that is why the replicated theory is RS — which is massless, and the glassy phase is unstable for any infinitesimal magnetic field, i.e. there is no AT line. A schematic picture of the two scenarios on the temperature-magnetic field plane is presented in Fig. 1. The phase boundary lies along the temperature axis in the droplet case, a zero-temperature fixed point governing its behaviour; the analogous attractive — and also zero-temperature — fixed point for the RSB scenario is shifted to a nonzero external field h_c. The other end of the phase boundary is, in both cases, the zero-field critical fixed point at T_c. Since the symmetry of the transition line — namely, an RS state with nonzero order parameter q, which is massless in the so-called replicon
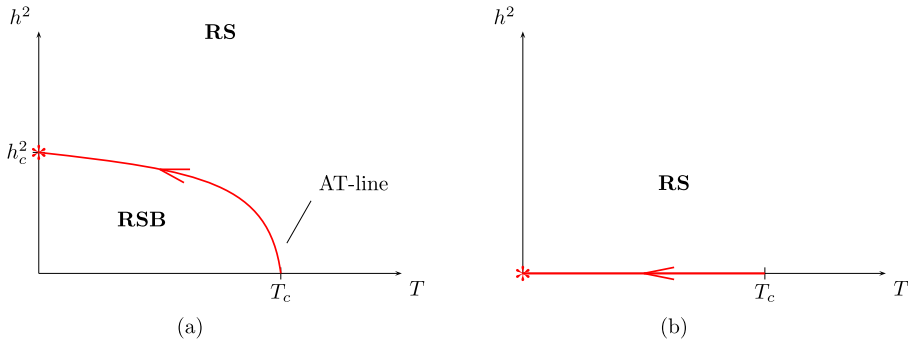
Fig. 1. Schematic phase diagrams for a d-dimensional Ising spin glass in the temperature-magnetic field plane. There is an RSB glassy phase in (a) bordered by the AT line. On the other hand, the glassy phase is RS in (b), and lies in the zero-field subspace. Both the AT line and the zero-field glassy phase are represented by the same generic replica symmetric field theory with massive longitudinal and massless replicon modes.

sector, while massive in the longitudinal one — is the same (notwithstanding the fact that the AT line proceeds in nonzero magnetic field), the two renormalization group (RG) pictures can be studied in a common field theory. This is the generic replica symmetric field theory elaborated in Refs. [9,15]. The vicinity of the (hypothetical) zero temperature fixed point can be studied in this field theory by assuming a hard (practically infinite) longitudinal mass, thus projecting the theory into the replicon sector. This was done decades ago by Bray and Roberts [16], who found a stable Gaussian fixed point for $d > 6$, whereas it was impossible to find any physically relevant and stable fixed point for $d < 6$. This was later interpreted [6,14] as a sign that the AT line disappears below six dimensions, and the droplet scenario takes over. This is, however, a faulty argument, since — as we have explained above — the RG equations (those for instance of Ref. [16]) are not specific to the low temperature behaviour of the AT line. An effort to understand the crossover from the zero-field critical fixed point to the zero-temperature one was made in Ref. [17], where the whole set of RG equations was derived in a first order perturbative renormalization. (The Bray–Roberts equations are naturally included there.) The runaway flows found were discussed in details in [7], and it was argued in this reference that the RG scheme used could not be expected to detect a zero-temperature fixed point in epsilon expansion. But again, the lack of a fixed point with infinite longitudinal mass *in the RG equations valid around the critical point* is not specific to spin glasses, and this property cannot distinguish between the two rival spin glass theories.

In a recent paper [6], Moore and Bray suggest a proof that RSB disappears when six dimensions is approached form above. They take the $d \to 6^+$ limit of known first order results, using RG arguments, for x_1 (the breakpoint of the order parameter function) and the AT line, and find both going to zero. We reproduce their results in a more complete RG scheme in Section 2, and show what is the fundamental flaw in their argument. At this point, the reader is advised to jump to Fig. 2(b) in Section 6 where x_1 is plotted against dimension along with the so-called scaling variable, which is effectively the relative error of the approximation. The breakpoint x_1 is monotonically increasing for decreasing dimension as long as the scaling variable is small. This is the range where the approximation is valid! However at around $d \approx 6.1$, the scaling variable starts to steeply increase (and actually goes to 1 for $d \to 6$), simultaneously x_1 suddenly changes its behaviour, and falls to zero: this is the effect (and a similar scenario for the AT line) that has been found in [6], but it must be clear that these results fall outside the range of validity of the approx-

imate RG equations. As a matter of fact, x_1 can be calculated directly in $d = 6$ (Section 3), its value is shown as the horizontal line in Fig. 2(b): it is visibly an extrapolation of the curve from the range where the approximation is good. (In fact, it is an old wisdom of the RG theories that the upper critical dimension requires special care.) There is only one case where the arguments of Ref. [6] are correct [and interestingly enough, this is admitted there below Eq. (18) of that reference], namely just at criticality. But that yields only the trivial results for the $d = 6$ system: x_1 is zero for $T = T_c$, and the AT line starts at the origin, i.e. at $T = T_c$ and $h^2 = 0$, and does not say anything about the disappearance of RSB.[1]

The outline of the paper is as follows: Section 2 is devoted to the study of the dimensional regime $6 < d < 8$, although the perturbative results of Section 2.1 are extensively used in later sections too. In Section 3, the renormalization group ideas are specifically applied to the $d = 6$ case, simply following the lines explained in classical RG textbooks (see, for instance, [18]). The breakpoint x_1 and the AT line are calculated at the upper critical dimension, both displaying logarithmic temperature corrections. A method for expanding a general (except that replica equivalence is assumed) RSB theory around the RS one is presented in Section 4, and applied to the ultrametric case. By this method, quantities of the RSB theory, like x_1, can be expressed in terms of vertices of the RS theory. In the next section, Section 5, we return to our original program, and study the case $d < 6$: generic RG arguments are presented, and the calculation of x_1 and the AT line in ϵ-expansion is performed. A new feature emerges below six dimensions, namely x_1 becomes nonzero and universal at criticality. In the last section, Section 6, special examples, both for x_1 and the AT line, are used to conclude that RSB escalates both in the regime above and below six dimensions.

2. Formulation of the spin glass problem for $6 < d < 8$

The simplest replicated field theory corresponding to the Ising spin glass in zero external magnetic field and below $d = 8$ has two bare parameters defining the model: τ (measuring the distance from criticality and w (the only bare cubic coupling compatible with the symmetrical — paramagnetic — state). Its Lagrangian is

$$\mathcal{L} = \frac{1}{2} \sum_{\mathbf{p}} \left(\frac{1}{2} p^2 + \bar{m} \right) \sum_{\alpha\beta} \phi_{\mathbf{p}}^{\alpha\beta} \phi_{-\mathbf{p}}^{\alpha\beta} - \frac{1}{6N^{1/2}} w \sum_{\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3}' \sum_{\alpha\beta\gamma} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\beta\gamma} \phi_{\mathbf{p}_3}^{\gamma\alpha} \tag{1}$$

where the bare mass $\bar{m} = \bar{m}_c - \tau$, and the critical mass has been presented in the literature several times in leading order of the loop expansion:

$$\bar{m}_c = \frac{1}{2}(n-2)w^2 \frac{1}{N} \sum_{\mathbf{p}} \frac{1}{p^4}.$$

In this $n(n-1)/2$ component field theory the fluctuating fields are symmetric in the replica indices with zero diagonals: $\phi_{\mathbf{p}}^{\alpha\beta} = \phi_{\mathbf{p}}^{\beta\alpha}$ and $\phi_{\mathbf{p}}^{\alpha\alpha} = 0$, $\alpha, \beta = 1, \ldots, n$. [Momentum conservation is indicated by the primed summation. The number N of the Ising spins becomes infinite in the thermodynamic limit, rendering summations to integrals over the continuum of momenta in the diagrams of the perturbative expansion. A momentum cutoff Λ is always understood to block

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[1] Somewhat surprisingly, [6] neglects discussing and even citing Ref. [7], where the AT line is followed up from mean field to $d \lesssim 6$. Section 5.3 reconsiders and confirms the existence of an AT line below six dimensions.

ultraviolet divergences, although it can be (and will be) absorbed into the definition of different quantities.] The replica number n goes to zero in the spin glass limit.

2.1. Perturbative results

We are now going to recollect several results for the replica symmetric (RS) spin glass phase — see Refs. [9,15,17,19] — which are needed for the following discussion. Due to the severe technical difficulties, only one-loop calculations have been accomplished ($\epsilon \equiv 6 - d$ and $n = 0$).

- RS order parameter q, i.e. the equation of state:

$$\frac{wq}{\tau} = 1 - 2w^2 \tau^{|\epsilon|/2} \frac{1}{N} \sum_{\mathbf{p}}^{\frac{\Lambda}{\sqrt{\tau}}} \frac{p^2 - 2}{p^4(p^2 + 2)^2} + \frac{1}{2} w\tau^{-2} h^2. \tag{2}$$

(The last term with the external magnetic field h has been included here for later reference. At the moment, it is to be considered as zero.) We can use $wq = \tau$ in the one-loop diagrams, and after rescaling the momentum as $p \to p/\sqrt{\tau}$, two different propagators remain: the replicon (p^{-2}) and the longitudinal [$(p^2 + 2)^{-1}$] ones. To make the formulae for the one-loop vertices more transparent, it is useful to introduce a common notation $I_{...}$ for the occurring integrals, as is illustrated below:

$$I_{RRLL} \equiv \frac{1}{N} \sum_{\mathbf{p}}^{\frac{\Lambda}{\sqrt{\tau}}} \frac{1}{p^4(p^2 + 2)^2} = \int^{\frac{\Lambda}{\sqrt{\tau}}} \frac{d^d p}{(2\pi)^d} \frac{1}{p^4(p^2 + 2)^2} = K_d \int^{\frac{\Lambda}{\sqrt{\tau}}} \frac{dp\, p^{-1+d}}{p^4(p^2 + 2)^2}.$$

- The replicon mass:

$$\Gamma_R = 2m_1 = -2\tau + 2wq + 4w^2 \tau^{1+|\epsilon|/2}(4I_{RLL} - 3I_{RRL}). \tag{3}$$

- The basic cubic vertex of the $\mathrm{Tr}\,\phi^3$ operator:

$$w_1 = w + 2w^3 \tau^{|\epsilon|/2}(-8I_{RRL} + 7I_{RRR} - 14I_{RRLL} - 8I_{RLLL}). \tag{4}$$

- The quartic vertex of $\phi^{\alpha\beta 4}$:

$$u_2 = 24w^4 \tau^{-1+|\epsilon|/2} I_{RRLL}. \tag{5}$$

In fact, this last result is new. Details of the somewhat lengthy calculation of the replicon-type quartic vertices will be published later.

2.2. Simple two-parameter renormalization group

An extensive renormalization group (RG) study of the generic RS glassy phase was published in Ref. [17]. When close to the Gaussian fixed point,[2] i.e. $w \ll 1$ and $\tau \ll 1$, and only infinitesimally breaking the high-temperature (paramagnetic) symmetry of the system, we have the following simple two-parameter RG flow-equations:

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[2] From now on, we redefine the parameters by suitably absorbing the geometrical factor K_d and Λ: $\tau/\Lambda^2 \to \tau$, $w^2 K_d \Lambda^{|\epsilon|} \to w^2$ and $h^2 K_d^{-1/2} \Lambda^{-4-|\epsilon|/2} \to h^2$.

$$\dot{w}^2 = -|\epsilon|w^2 - 2w^4,$$
$$\dot{\tau} = \left(2 - \frac{10}{3}w^2\right)\tau. \tag{6}$$

Physical quantities take simple scaling forms when, instead of w and τ, they are expressed in terms of the nonlinear scaling fields $\tilde{w}$ and r defined by:

$$\dot{\tilde{w}}^2 = -|\epsilon|\tilde{w}^2,$$
$$\dot{r} = 2r. \tag{7}$$

A straightforward calculation provides:

$$w^2 = \tilde{w}^2\left(1 - 2\frac{\tilde{w}^2}{|\epsilon|}\right)^{-1},$$
$$\tau = r\left(1 - 2\frac{\tilde{w}^2}{|\epsilon|}\right)^{-5/3}. \tag{8}$$

We are now going to compute the quantities q, Γ_R, w_1 and u_2 by the RG in terms of $\tilde{w}^2$ and r. In this way, we can get more general results when approaching dimension six from above as compared with the perturbative computation: now we may have $|\epsilon| \ll w^2 \ll 1$, although the scaling variable $\tilde{w}^2 r^{|\epsilon|/2}$ must be small:

$$\tilde{w}^2 r^{|\epsilon|/2} \ll |\epsilon|, \quad \text{even when } |\epsilon| \ll w^2.$$

• The renormalization flow equation for q is

$$\dot{q} = \left(2 + \frac{|\epsilon|}{2} + \frac{\eta_L}{2}\right)q \tag{9}$$

with $\eta_L = \eta_R = -\frac{2}{3}w^2$ in this approximation. It can be solved by using Eqs. (7) and (8):

$$q = r^{1+\frac{|\epsilon|}{4}}\hat{q}\left(\tilde{w}^2 r^{|\epsilon|/2}\right)\left(1 - 2\frac{\tilde{w}^2}{|\epsilon|}\right)^{-1/6}, \tag{10}$$

and a comparison with (2) makes it possible — after some manipulations — to get the leading terms of the scaling function:

$$\hat{q}(x) = \frac{1}{\sqrt{x}}(1 + Cx + \cdots),$$
$$\text{with the constant } C = 2^{1+\frac{|\epsilon|}{2}}\Gamma\left(1 + \frac{|\epsilon|}{2}\right)\Gamma\left(1 - \frac{|\epsilon|}{2}\right)\left(\frac{1}{|\epsilon|} + 1\right). \tag{11}$$

• The replicon mass evolves under renormalization as

$$\dot{\Gamma}_R = (2 - \eta_R)\Gamma_R = \left(2 + \frac{2}{3}w^2\right)\Gamma_R, \tag{12}$$

with the solution

$$\Gamma_R = r\hat{\Gamma}_R\left(\tilde{w}^2 r^{|\epsilon|/2}\right)\left(1 - 2\frac{\tilde{w}^2}{|\epsilon|}\right)^{1/3}. \tag{13}$$

Substituting q in Eq. (3) by τ from (2) provides:

$$\Gamma_R = -16w^2\tau^{1+\frac{|\epsilon|}{2}} I_{RRLL} + w\tau^{-1}h^2. \tag{14}$$

Keeping in mind that (14) is valid for $\tilde{w}^2 \approx w^2 \ll |\epsilon|$ and h^2 is zero at the moment, it is straightforward to derive the scaling function in Eq. (13):

$$\hat{\Gamma}_R(x) = C'x + \cdots, \quad \text{with } C' = -2^{2+\frac{|\epsilon|}{2}} \Gamma\left(1+\frac{|\epsilon|}{2}\right)\Gamma\left(1-\frac{|\epsilon|}{2}\right). \tag{15}$$

• As for w_1, we have

$$\dot{w}_2 = \left(-\frac{|\epsilon|}{2} - \frac{3}{2}\eta_R\right)w_1 = \left(-\frac{|\epsilon|}{2} + w^2\right)w_1 \tag{16}$$

and

$$w_1 = r^{-\frac{|\epsilon|}{4}} \hat{w}_1\left(\tilde{w}^2 r^{|\epsilon|/2}\right)\left(1 - 2\frac{\tilde{w}^2}{|\epsilon|}\right)^{1/2}. \tag{17}$$

Comparing (4) and (17) yields

$$\hat{w}_1(x) = \sqrt{x}\left(1 + C''x + \cdots\right),$$
$$\text{with } C'' = 2^{\frac{|\epsilon|}{2}} \Gamma\left(1+\frac{|\epsilon|}{2}\right)\Gamma\left(1-\frac{|\epsilon|}{2}\right)\left(\frac{16}{|\epsilon|} - 9 - |\epsilon|\right). \tag{18}$$

• Finally, from the flow

$$\dot{u}_2 = (-2 - |\epsilon| - 2\eta_R)u_2 = \left(-2 - |\epsilon| + \frac{4}{3}w^2\right)u_2 \tag{19}$$

follows the scaling form of the most important quartic vertex:

$$u_2 = r^{-1-\frac{|\epsilon|}{2}} \hat{u}_2\left(\tilde{w}^2 r^{|\epsilon|/2}\right)\left(1 - 2\frac{\tilde{w}^2}{|\epsilon|}\right)^{2/3}. \tag{20}$$

From (5) and (20) results [see also (15)]

$$\hat{u}_2(x) = -\frac{3}{2}C'x^2 + \cdots. \tag{21}$$

2.3. The calculation of x_1 and the Almeida–Thouless line

The leading contribution to the breakpoint of the order parameter function $q(x)$ is derived in Section 4, and has the simple form [see (42) and the more general considerations in that section about getting x_1 on the basis of the generic RS field theory]:

$$x_1 = \frac{u_2}{w_1}q.$$

Inserting (10), (17) and (20), the scaling equation of x_1 follows:

$$x_1 = \hat{x}_1\left(\tilde{w}^2 r^{|\epsilon|/2}\right), \quad \text{with } \hat{x}_1(\ldots) = \frac{\hat{u}_2(\ldots)}{\hat{w}_1(\ldots)}\hat{q}(\ldots).$$

By the help of Eqs. (11), (18), (21) and (15), we can conclude

$$x_1 = 6 \times 2^{\frac{|\epsilon|}{2}} \Gamma\left(1+\frac{|\epsilon|}{2}\right)\Gamma\left(1-\frac{|\epsilon|}{2}\right)\tilde{w}^2 r^{|\epsilon|/2} + \cdots. \tag{22}$$

Inverting (8), x_1 can be expressed by the original bare coupling w:

$$x_1 \sim \frac{w^2}{1 + 2\frac{w^2}{|\epsilon|}} r^{|\epsilon|/2}. \tag{23}$$

This equation agrees with Eq. (21) of Ref. [6].[3] The range of applicability of the above equation:

$$w^2, r \ll 1, \qquad 0 < |\epsilon| < 2 \quad \text{and (most importantly)} \quad \tilde{w}^2 r^{|\epsilon|/2} \ll |\epsilon|. \tag{24}$$

If we fix the system's bare coupling w and approach six dimensions, then $\tilde{w}^2 \to |\epsilon|/2$ and $x_1 \sim |\epsilon| r^{|\epsilon|/2}$. This behaviour was interpreted by the authors of Ref. [6] as the sign of the end of RSB at six dimensions: a vanishing x_1 is consistent with RS. But, as Eq. (24) clearly shows, in this limit r must go to zero,[4] i.e. the breakpoint disappears at the critical surface in six dimensions — a property valid also for $d > 6$ (but, as we will see later, not for $d < 6$).[5] In the next section we will show that below the critical surface $x_1 > 0$ and has a logarithmic temperature dependence at exactly six dimensions.

We now turn to the problem of the Almeida–Thouless line. The introduction of a magnetic field h^2 involves a new nonlinear scaling field $\tilde{h}^2$ with

$$\dot{\tilde{h}}^2 = \left(4 + \frac{|\epsilon|}{2}\right)\tilde{h}^2.$$

Eq. (13) remains valid, but the scaling function $\hat{\Gamma}_R$ has now two arguments: $x = \tilde{w}^2 r^{|\epsilon|/2}$ and $y = \tilde{h}^2 r^{-2-|\epsilon|/4}$. Realizing that the replicon mass starts at one-loop order, the bare parameters in (14) can be replaced by their corresponding nonlinear scaling fields, making it possible to read off the scaling function:

$$\hat{\Gamma}_R(x, y) = C'x + \sqrt{x}\, y;$$

see also (15). The vanishing replicon mass defines the AT line, i.e. $y = -C'\sqrt{x}$ providing

$$\tilde{h}^2 = -C'\tilde{w} r^{2 + \frac{|\epsilon|}{2}}. \tag{25}$$

The connection between h^2 and $\tilde{h}^2$ may be found from the flow equation

$$\dot{h}^2 = \left(4 + \frac{|\epsilon|}{2} - \frac{\eta_L}{2}\right)h^2 = \left(4 + \frac{|\epsilon|}{2} + \frac{1}{3}w^2\right)h^2, \tag{26}$$

with the solution [see also (7) and (8)]:

$$h^2 = \tilde{h}^2\left(1 - 2\frac{\tilde{w}^2}{|\epsilon|}\right)^{1/6}. \tag{27}$$

It is useful to display the AT line (25) in the original bare parameters by Eqs. (8) and (27):

$$h^2 = -C' \frac{w}{(1 + 2\frac{w^2}{|\epsilon|})^{4 + \frac{5}{6}|\epsilon|}} \tau^{2 + \frac{|\epsilon|}{2}}. \tag{28}$$

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[3] $|r(0)| = |r|$ in that paper is what we call τ here, whereas $w(0) = w$ agrees with our notation for the bare cubic coupling.

[4] That point has been noticed in Ref. [6], but was completely misinterpreted. We will return to this problem in Section 6; see the first row of Eq. (59) showing the impossibility of the limit $|\epsilon| \to 0$ in this approximation.

[5] The multiplicative factor $|\epsilon|$ in x_1 has its origin in the termination of the definition of the nonlinear scaling field $\tilde{w}$ in $d = 6$. This is a feature of the RS renormalization group, and is not related to the problem of replica symmetry breaking.

This equation is identical with Eq. (15) of Ref. [6], and the $|\epsilon|^4$ factor, arising when $|\epsilon| \to 0$ while fixing w, led those authors to conclude that the AT line disappears in six dimensions. But, again, Eq. (24) and the discussion below it shows that this limit provides results only on the critical surface (τ and r zero), and it informs us only about the trivial fact that the AT line starts at the origin of the τ, h^2 plain.

3. At the upper critical dimension: $d = 6$

As can be seen from the previous section, knowledge about the six-dimensional system cannot be gained from the RG results in the $d \gtrsim 6$ case. The fundamental reason for that is the impossibility to linearize the RG flow equations at exactly an upper critical dimension. Therefore, the scaling field $\tilde{w}$ is not defined for $d = 6$, and we keep w (although r and $\tilde{h}^2$ are still meaningful). The RG flow (6) is now:

$$\dot{w}^2 = -2w^4,$$
$$\dot{\tau} = \left(2 - \frac{10}{3}w^2\right)\tau. \tag{29}$$

The connection between τ and r becomes [instead of (8)]:

$$\tau = rw^{\frac{10}{3}}, \tag{30}$$

and the scaling variable with zero scaling dimension is now (instead of $\tilde{w}^2 r^{|\epsilon|/2}$):

$$\frac{w^2}{1 - w^2 \ln r},$$

which can be easily checked by Eq. (29) and the nonlinear scaling field property $\dot{r} = 2r$.

3.1. The calculation of x_1

The renormalization group flow equations for the three relevant physical quantities q, w_1 and u_2 are as follows:

$$\dot{q} = \left(2 - \frac{1}{3}w^2\right)q,$$
$$\dot{w}_2 = w^2 w_1,$$
$$\dot{u}_2 = \left(-2 + \frac{4}{3}w^2\right)u_2.$$

They all have the same form, and their solutions are easily found in scaling form.

- The RS order parameter:

$$q = w^{\frac{1}{3}}r\hat{q}\left(\frac{w^2}{1 - w^2 \ln r}\right),$$
$$\hat{q}(x) = x\left[1 + (2 + \ln 2)x + \frac{5}{3}x\ln x + \cdots\right]. \tag{31}$$

The scaling function $\hat{q}(x)$ has been obtained by evaluating (2) in $d = 6$ (in zero magnetic field at the moment) and using the connection between τ and r in (30).

- The cubic vertex w_1 in six dimensions:

$$w_1 = w^{-1} \hat{w}_1 \left(\frac{w^2}{1 - w^2 \ln r} \right),$$

$$\hat{w}_1 = x \left[1 + \left(-\frac{39}{2} + 8 \ln 2 - 7 \ln n \right) x + \frac{5}{3} x \ln x + \cdots \right]. \tag{32}$$

Eqs. (4) and (30) has been used to get the scaling function. One important remark is appropriate here. The term with the logarithm of the replica number, $\ln n$, comes from I_{RRR} in (4), and is a prominent example of the severe infrared divergences caused by the replicon propagator. Similar contributions enter in higher order vertices, such as I_{RRRR} in the quartic vertex belonging to the operator $\mathrm{Tr}\,\phi^4$. This is a clear indication — beside the instability of the replicon mode — that the replica symmetric theory is ill-defined in the spin glass limit. In fact, these infrared divergent terms can be resummed when we build up the RSB theory on the basis of the RS one, as explained in Section 4. What is gained in this resummation, after setting n to zero, is the small mass regime of the RSB solution which effectively acts as an infrared cutoff. It must be stressed that without this resummation, the theory is infrared divergent in any arbitrarily high dimension.

- As for the quartic vertex u_2, its scaling form and the leading term of the scaling function are [see (5) and (30)]:

$$u_2 = w^{-\frac{4}{3}} r^{-1} \hat{u}_2 \left(\frac{w^2}{1 - w^2 \ln r} \right),$$

$$\hat{u}_2(x) = 6x + \cdots. \tag{33}$$

By Eqs. (31), (32) and (33) x_1 turns out to be a function of the scaling variable, as it must be:

$$x_1 = \frac{u_2}{w_1} q = \hat{x}_1 \left(\frac{w^2}{1 - w^2 \ln r} \right) \quad \text{with } \hat{x}_1(\ldots) = \frac{\hat{u}_2(\ldots)}{\hat{w}_1(\ldots)} \hat{q}(\ldots).$$

The leading order of the scaling function is simply $\hat{x}_1(x) = 6x + \cdots$, providing one of our basic results

$$x_1 = 6 \left(\frac{w^2}{1 - w^2 \ln r} \right) + \cdots; \quad w, r \ll 1 \text{ and } r = \tau w^{-\frac{10}{3}}, \quad d = 6. \tag{34}$$

It is clear from the above equation that x_1 is zero at criticality ($r = \tau = 0$), and for fixed w the approach to zero is logarithmic:

$$x_1 = 6 |\ln r|^{-1} + \cdots; \quad r, \tau \to 0 \text{ and } w \text{ fixed}, \quad d = 6.$$

3.2. Almeida–Thouless line in six dimensions

The flow equation for the replicon mass is unchanged as compared with the $d > 6$ case, and is given by Eq. (12). The nonlinear scaling field corresponding to the external magnetic field satisfies $\dot{\tilde{h}}^2 = 4\tilde{h}^2$, therefore the second variable with zero scaling dimension is $\tilde{h}^2/r^2$. Straightforward considerations lead us to

$$\Gamma_R = w^{-\frac{2}{3}} r \hat{\Gamma}_R \left(\frac{w^2}{1 - w^2 \ln r}, \frac{\tilde{h}^2}{r^2} \right). \tag{35}$$

The evolution of the "bare" magnetic field, i.e. $\dot{h}^2 = (4 + \frac{1}{3}w^2)h^2$ [see (26)] and (29) yield

$$h^2 = \tilde{h}^2 w^{-\frac{1}{3}}. \tag{36}$$

Evaluating Eq. (14) at $d = 6$, and replacing the bare parameters τ and h^2 by r and $\tilde{h}^2$ according to (30) and (36), respectively, makes it possible to read off the scaling function in leading order:

$$\hat{\Gamma}_R(x, y) = \frac{1}{x}\left(-4x^4 + y + \cdots\right).$$

From its zero, the AT line is obtained as follows:

$$\tilde{h}^2 = 4r^2\left(\frac{w^2}{1 - w^2 \ln r}\right)^4 + \cdots; \quad w, r \ll 1 \text{ and } r = \tau w^{-\frac{10}{3}},$$

$$\tilde{h}^2 = h^2 w^{\frac{1}{3}}; \quad d = 6. \tag{37}$$

For a given cubic coupling w, the magnetic field vs. temperature relationship for the boundary of the RS phase when approaching the critical point becomes:

$$\tilde{h}^2 = 4r^2|\ln r|^{-4} + \cdots; \quad r, \tau \to 0 \text{ and } w \text{ fixed}, \quad d = 6.$$

4. Formulation of replica symmetry breaking on the basis of the generic replica symmetric theory

The considerations in this section are quite general and, therefore, the paramagnetic system (i.e. an RS system with zero order parameter) must be represented — instead of the simple case of (1) which is sufficient around $d = 6$ — by a model which includes all the invariants compatible with its higher symmetry [20]. The replicated field theory is now defined by the Lagrangian $\mathcal{L}$ of the symmetrical (high-temperature and zero-field) theory:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\sum_{\mathbf{p}}\left(\frac{1}{2}p^2 + \bar{m}_1\right)\sum_{\alpha\beta}\phi_{\mathbf{p}}^{\alpha\beta}\phi_{-\mathbf{p}}^{\alpha\beta} - \frac{1}{6N^{1/2}}\sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3}'\bar{w}_1\sum_{\alpha\beta\gamma}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\beta\gamma}\phi_{\mathbf{p}_3}^{\gamma\alpha} \\
&\quad - \frac{1}{24N}\sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4}'\left(\bar{u}_1\sum_{\alpha\beta\gamma\delta}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\beta\gamma}\phi_{\mathbf{p}_3}^{\gamma\delta}\phi_{\mathbf{p}_4}^{\delta\alpha} + \bar{u}_2\sum_{\alpha\beta}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\beta}\phi_{\mathbf{p}_3}^{\alpha\beta}\phi_{\mathbf{p}_4}^{\alpha\beta}\right. \\
&\quad \left. + \bar{u}_3\sum_{\alpha\beta\gamma}\phi_{\mathbf{p}_1}^{\alpha\gamma}\phi_{\mathbf{p}_2}^{\alpha\gamma}\phi_{\mathbf{p}_3}^{\beta\gamma}\phi_{\mathbf{p}_4}^{\beta\gamma} + \bar{u}_4\sum_{\alpha\beta\gamma\delta}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\beta}\phi_{\mathbf{p}_3}^{\gamma\delta}\phi_{\mathbf{p}_4}^{\gamma\delta}\right) \\
&\quad - \frac{1}{120N^{3/2}}\sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4\mathbf{p}_5}'\left(\bar{v}_1\sum_{\alpha\beta\gamma\delta\mu}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\beta\gamma}\phi_{\mathbf{p}_3}^{\gamma\delta}\phi_{\mathbf{p}_4}^{\delta\mu}\phi_{\mathbf{p}_5}^{\mu\alpha} + \bar{v}_2\sum_{\alpha\beta\gamma}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\beta}\phi_{\mathbf{p}_3}^{\alpha\beta}\phi_{\mathbf{p}_4}^{\alpha\gamma}\phi_{\mathbf{p}_5}^{\beta\gamma}\right. \\
&\quad \left. + \bar{v}_3\sum_{\alpha\beta\gamma\delta}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\beta\gamma}\phi_{\mathbf{p}_3}^{\gamma\alpha}\phi_{\mathbf{p}_4}^{\gamma\delta}\phi_{\mathbf{p}_5}^{\gamma\delta} + \bar{v}_4\sum_{\alpha\beta\gamma\mu\nu}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\beta\gamma}\phi_{\mathbf{p}_3}^{\gamma\alpha}\phi_{\mathbf{p}_4}^{\mu\nu}\phi_{\mathbf{p}_5}^{\mu\nu}\right) + \cdots \tag{38}
\end{aligned}$$

where the bare mass $\bar{m}_1 \equiv \bar{m} = \bar{m}_c - \tau$, with τ measuring the distance from criticality, has been also used in (1), and $\bar{w}_1 \equiv w$, momentum conservation is indicated by the primed summations). The fifth order invariants with the $\bar{v}$ bare couplings were also included here. In what follows, we use the same notation for an *exact* vertex (e.g. u_2) and its corresponding bare coupling ($\bar{u}_2$), the bar indicating always a bare quantity.

As explained in details in Appendix D of Ref. [9], the generic Legendre-transformed free energy can be expanded around the RS spin glass state with order parameter q; see (D.5) of this reference:

$$
\begin{aligned}
\frac{1}{N}&\mathcal{F}(q_{\alpha\beta})\\
&= \frac{1}{N}\mathcal{F}(q) + \frac{1}{2}\left[m_1 \sum_{\alpha\beta}(q_{\alpha\beta}-q)^2 + m_2 \sum_{\alpha\beta\gamma}(q_{\alpha\gamma}-q)(q_{\beta\gamma}-q)\right.\\
&\quad\left. + m_3 \sum_{\alpha\beta\gamma\delta}(q_{\alpha\beta}-q)(q_{\gamma\delta}-q)\right]\\
&\quad - \frac{1}{6}\left[w_1 \sum_{\alpha\beta\gamma}(q_{\alpha\beta}-q)(q_{\beta\gamma}-q)(q_{\gamma\alpha}-q) + w_2 \sum_{\alpha\beta}(q_{\alpha\beta}-q)^3\right.\\
&\quad\left. + w_3 \sum_{\alpha\beta\gamma}(q_{\alpha\beta}-q)^2(q_{\alpha\gamma}-q) + \cdots\right]\\
&\quad - \frac{1}{24}\left[u_1 \sum_{\alpha\beta\gamma\delta}(q_{\alpha\beta}-q)(q_{\beta\gamma}-q)(q_{\gamma\delta}-q)(q_{\delta\alpha}-q) + u_2 \sum_{\alpha\beta}(q_{\alpha\beta}-q)^4 + \cdots\right]\\
&\quad - \frac{1}{120}\left[v_1 \sum_{\alpha\beta\gamma\delta\mu}(q_{\alpha\beta}-q)(q_{\beta\gamma}-q)(q_{\gamma\delta}-q)(q_{\delta\mu}-q)(q_{\mu\alpha}-q) + \cdots\right] + \cdots.
\end{aligned}
$$
(39)

In zero external field $\mathcal{F}(q_{\alpha\beta})$ has the same symmetry as $\mathcal{L}$ of Eq. (38) — which is higher than that of a generic RS system — even when $T < T_c$, and using this symmetry, a set of equations can be found between the exact vertices of the generic RS theory (see Refs. [9,20]). The most effective way to get the required vertex relationships is demanding that invariants incompatible with the symmetrical theory, e.g. $\sum_{\alpha\beta} q_{\alpha\beta}^3$, must finally disappear from (39). In this manner, all the vertices of the lower symmetry: m_2, m_3; $w_2, \ldots, w_8$; $u_5, \ldots, u_{23}$; etc. (see Appendix A of Ref. [9] for the classification of cubic and quartic vertices) and, as a bonus, m_1 can be expressed in terms of w_1, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, and higher order symmetrical vertices.[6] We than have

$$
m_1 = \frac{1}{2}nw_1 q + \frac{1}{6}(n^2 u_1 - 2u_2)q^2 + \frac{1}{24}n(n^2 v_1 - 2v_2)q^3 + \cdots,
$$

$$
m_2 = -w_1 q - \frac{1}{3}(nu_1 + u_3)q^2 + \frac{1}{60}\left[5n(3n^2 - 5n + 1)v_1 + 2v_2 - 4nv_3\right]q^3 + \cdots,
$$

$$
m_3 = -\frac{1}{6}(u_1 + 2u_4)q^2 - \frac{1}{60}\left[5(5n - 4)v_1 + 2v_3 + 6nv_4\right]q^3 + \cdots,
$$

and furthermore

$$
w_2 = u_2 q + \frac{1}{20}nv_2 q^2 + \cdots,
$$

$$
w_3 = u_3 q + \frac{1}{10}(3v_2 + nv_3)q^2 + \cdots,
$$

---

[6] A vertex is called symmetrical if it is nonzero in the zero order parameter RS system.

$$w_4 = u_4 q + \frac{1}{20}(v_3 + 3nv_4)q^2 + \cdots,$$

$$w_5 = u_1 q + \frac{1}{20}(5nv_1 + 4v_3)q^2 + \cdots,$$

$$w_6 = \frac{1}{10}v_3 q^2 + \cdots,$$

$$w_7 = \frac{1}{40}(10v_1 + 12v_4)q^2 + \cdots,$$

$$w_8 = O(q^3).$$

Of the quartic vertices, only those are listed below which are required up to the order of the present calculation:

$$u_5 = \frac{3}{5}v_2 q + \cdots, \qquad u_6 = \frac{2}{5}v_3 q + \cdots, \qquad u_7 = \frac{2}{5}v_4 q + \cdots,$$

$$u_8 = \frac{2}{5}v_2 q + \cdots, \qquad u_{10} = \frac{1}{5}v_3 q + \cdots, \qquad u_{11} = \frac{2}{5}v_3 q + \cdots,$$

$$u_{14} = \frac{3}{5}v_4 q + \cdots, \qquad u_{16} = v_1 q + \cdots.$$

By exploiting these expressions, the free energy functional in Eq. (39) can now be written (omitting an additive term depending only on q):

$$\frac{1}{N}\mathcal{F}(q_{\alpha\beta}) = \frac{1}{4}Mq\sum_{\alpha\beta}q_{\alpha\beta}^2 - \frac{1}{6}W\sum_{\alpha\beta\gamma}q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha}$$

$$- \frac{1}{24}\left[(u_1 + v_1 q + \cdots)\sum_{\alpha\beta\gamma\delta}q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\delta}q_{\delta\alpha} + \left(u_2 + \frac{2}{5}v_2 q + \cdots\right)\sum_{\alpha\beta}q_{\alpha\beta}^4\right.$$

$$\left. + \left(u_3 + \frac{3}{5}v_3 q + \cdots\right)\sum_{\alpha\beta\gamma}q_{\alpha\gamma}^2 q_{\beta\gamma}^2 + \left(u_4 + \frac{3}{5}v_4 q + \cdots\right)\left(\sum_{\alpha\beta}q_{\alpha\beta}^2\right)^2\right]$$

$$- \frac{1}{120}\left[(v_1 + \cdots)\sum_{\alpha\beta\gamma\delta\mu}q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\delta}q_{\delta\mu}q_{\mu\alpha} + (v_2 + \cdots)\sum_{\alpha\beta\gamma}q_{\alpha\beta}^3 q_{\alpha\gamma}q_{\beta\gamma}\right.$$

$$\left. + (v_3 + \cdots)\sum_{\alpha\beta\gamma\delta}q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha}q_{\gamma\delta}^2 + (v_4 + \cdots)\left(\sum_{\alpha\beta\gamma}q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha}\right)\left(\sum_{\alpha\beta}q_{\alpha\beta}^2\right)\right]$$

$$- \cdots,$$

with the following notations for M and W:

$$M \equiv (n-2)w_1 + \frac{1}{3}\left[(n^2 - 3)u_1 + u_2 + (n-1)\tilde{u}_3\right]q$$

$$+ \frac{1}{60}\left[5(n^3 - 4)v_1 - 2(2n - 8)v_2 + 2(n-1)(n+4)\tilde{v}_3\right]q^2 + \cdots,$$

$$W \equiv w_1 + u_1 q + \frac{1}{20}\left[10v_1 - 3v_2 - (n-1)\tilde{v}_3\right]q^2 + \cdots.$$

The tilded vertices $\tilde{u}_3 \equiv u_3 + nu_4$ and $\tilde{v}_3 \equiv v_3 + nv_4$ were introduced here; in fact, only these combinations will enter the equation of state.

Stationarity of the free energy functional provides the equation of state:

$$
\begin{aligned}
0 = {} & Mqq_{\alpha\beta} - W(q^2)_{\alpha\beta} - \frac{1}{3}\left[(u_1 + v_1 q + \cdots)(q^3)_{\alpha\beta} + \left(u_2 + \frac{2}{5}v_2 q + \cdots\right)q_{\alpha\beta}^3 \right. \\
& + \left(\tilde{u}_3 + \frac{3}{5}\tilde{v}_3 q + \cdots\right)(q^2)_{\alpha\alpha}q_{\alpha\beta}\right] - \frac{1}{60}\left\{5(v_1 + \cdots)(q^4)_{\alpha\beta}\right. \\
& + (v_2 + \cdots)\left[3(q^2)_{\alpha\beta}q_{\alpha\beta}^2 + \sum_{\gamma}(q_{\alpha\gamma}^3 q_{\beta\gamma} + q_{\beta\gamma}^3 q_{\alpha\gamma})\right] \\
& + (\tilde{v}_3 + \cdots)\left[2(q^3)_{\alpha\alpha}q_{\alpha\beta} + 3(q^2)_{\alpha\alpha}(q^2)_{\alpha\beta}\right]\Big\} - \cdots.
\end{aligned}
\tag{40}
$$

Only replica equivalence was used in the derivation of this equation — $(q^2)_{\alpha\alpha}$, for instance, is independent of the replica number — otherwise it is quite general: it provides an RSB solution in terms of the RS order parameter q (which measures the distance from criticality now), and of the exact RS vertices. It can equally be used in any regime where some kind of perturbation theory is valid.

We now turn to the case of infinite step, ultrametrically organized RSB. The small parameter making possible a perturbative treatment is x_1, the breakpoint of the order parameter function: it is proportional to q in the SK model and for the field theory above 8 dimensions, to $q^{d/2-3}$ between 6 and 8 dimensions, whereas it is of order ϵ below 6 dimensions. $q(x)$, the order parameter function, has the form[7]:

$$
q(x) = q_1\left[r + x_1^2 \delta\bar{q}(r)\right], \quad \text{with } r \equiv x/x_1 \text{ and } \delta\bar{q}(1) = 0.
\tag{41}
$$

The contributions of the various vertices to Eq. (40) are listed below. The definition of the bilinear expression $\{\ldots; \ldots\}$ used extensively in that list is as follows:

$$
\begin{aligned}
\{f(r); g(r)\} \equiv {} & f(r)g(1) + f(1)g(r) - f(r)\int_r^1 du\, g(u) - g(r)\int_r^1 du\, f(u) - rf(r)g(r) \\
& - \int_0^r du\, f(u)g(u).
\end{aligned}
$$

- w_1:

$$
2(q_1 - q)r - x_1 q_1\left(r - \frac{1}{3}r^3\right) + 2x_1^2(q_1 - q)\delta\bar{q}(r) - 2x_1^3 q_1\{r; \delta\bar{q}(r)\} + O(x_1^4),
$$

- u_1:

$$
-(q_1 - q)^2 r + x_1(q_1 - q)q_1\left(r - \frac{1}{3}r^3\right) - \frac{1}{3}x_1^2 q_1^2\left(\frac{3}{4}r - \frac{1}{2}r^3 + \frac{3}{20}r^5\right) + O(x_1^4),
$$

- u_2:

$$
\frac{1}{3}q^2 r - \frac{1}{3}q_1^2 r^3 + \frac{1}{3}x_1^2 q^2 \delta\bar{q}(r) - x_1^2 q_1^2 r^2 \delta\bar{q}(r) + O(x_1^4),
$$

---

[7] We hope that the *ratio* r of x to x_1 introduced here cannot be confused with the temperature-like scaling field of previous sections.

- $\tilde{u}_3$:

$$\frac{1}{3}(q_1^2 - q^2)r - \frac{2}{9}x_1 q_1^2 r + \frac{1}{3}x_1^2(q_1^2 - q^2)\delta\bar{q}(r) - \frac{2}{3}x_1^3 q_1^2 r\{r; \delta\bar{q}(r)\}_{r=1}$$

$$- \frac{2}{9}x_1^3 q_1^2 \delta\bar{q}(r) + O(x_1^4),$$

- v_1:

$$\frac{1}{3}(q_1 - q)^3 r - \frac{1}{2}x_1(q_1 - q)^2 q_1\left(r - \frac{1}{3}r^3\right) + \frac{1}{3}x_1^2(q_1 - q)q_1^2\left(\frac{3}{4}r - \frac{1}{2}r^3 + \frac{3}{20}r^5\right)$$

$$- \frac{1}{12}x_1^3 q_1^3\left(\frac{1}{2}r - \frac{1}{2}r^3 + \frac{3}{10}r^5 - \frac{1}{14}r^7\right) + O(x_1^4),$$

- v_2:

$$-\frac{1}{5}(q_1 - q)q^2 r + \frac{2}{15}(q_1 - q)q_1^2 r^3 - \frac{1}{30}x_1 q_1^3\left(\frac{3}{4}r + \frac{1}{2}r^3 - \frac{9}{20}r^5\right)$$

$$+ \frac{3}{20}x_1 q^2 q_1\left(r - \frac{1}{3}r^3\right) - \frac{1}{20}x_1 q_1^3 r^2\left(r - \frac{1}{3}r^3\right) + \frac{1}{10}(q_1 - q)^2 qr$$

$$+ \frac{1}{30}(q_1 - q)^3 r - \frac{1}{5}x_1^2(q_1 - q)q_1^2(1 - 2r^2)\delta\bar{q}(r) - \frac{1}{10}x_1^3 q_1^3 r\left(r - \frac{1}{3}r^3\right)\delta\bar{q}(r)$$

$$+ \frac{3}{10}x_1^3 q_1^3\{r; \delta\bar{q}(r)\} - \frac{1}{10}x_1^3 q_1^3 r^2\{r; \delta\bar{q}(r)\} - \frac{1}{30}x_1^3 q_1^3\{r^3; \delta\bar{q}(r)\}$$

$$- \frac{1}{10}x_1^3 q_1^3\{r; r^2\delta\bar{q}(r)\} + O(x_1^4),$$

- $\tilde{v}_3$:

$$\left[-\frac{3}{10}(q_1 - q)^2 q_1 + \frac{2}{15}x_1(q_1 - q)q_1^2 - \frac{1}{75}x_1^2 q_1^3 + \frac{2}{15}(q_1 - q)^3\right]r$$

$$+ \frac{1}{20}x_1 q_1\left[2(q_1 - q)q_1 - \frac{2}{3}x_1 q_1^2 - (q_1 - q)^2\right]\left(r - \frac{1}{3}r^3\right) + O(x_1^4).$$

Inserting the above expressions into Eq. (40) and demanding that the coefficients of r and r^3 disappear, x_1 can be read off with some effort. It is best to give x_1 as the zero, $f(x_1) = 0$, of the following function:

$$f(x) \equiv \left[-\left(\frac{u_2}{w_1}q\right) + \frac{1}{2}\left(\frac{y_2}{w_1}q^3\right) + \cdots\right] + \left[1 - \frac{13}{60}\left(\frac{v_2}{w_1}q^2\right) + \cdots\right]x$$

$$+ \left[-\frac{1}{3} + \frac{1}{6}\left(\frac{u_1}{w_1}q\right) + \cdots\right]x^2 + \left[-\frac{1}{9} + \cdots\right]x^3 + \cdots.$$

The leading contribution is the well-known formula

$$x_1 = \frac{u_2}{w_1}q \tag{42}$$

which is used extensively throughout this paper. As a byproduct, the shift of q_1 from the RS order parameter is given by

$$q_1 - q = \frac{1}{3}x_1 q\left(1 + \frac{2}{3}x_1 + \cdots\right). \tag{43}$$

[To preserve consistency, a sixth order contribution $-\frac{1}{6!}y_2\sum_{\alpha\beta}(q_{\alpha\beta}-q)^6$ should have been included in the free energy expansion (39), as it enters the constant of $f(x)$ at the third order, i.e. at the highest order studied here.]

5. Below six dimensions

5.1. The renormalization group: fixed point and nonlinear scaling fields

In $d = 6 - \epsilon$ the Gaussian fixed point becomes unstable, and the zero-field spin glass transition is governed by the nontrivial one. Here we collect and present the available results for this fixed point (in the results for the fixed point below, a generic n is kept, although $n = 0$ is taken in the further parts of the section):

$$\bar{w}_1^{*\,2} \equiv w^{*\,2} = \frac{1}{2-n}\epsilon, \qquad \bar{m}_1^* \equiv \bar{m}^* = -\frac{2-n}{4}w^{*\,2}; \quad \text{see Refs. [21] and [17].}$$

Although they will not be used in this paper, the fixed point values of the quartic couplings (which — according to our knowledge — have not been published before) are also listed here:

$$\bar{u}_1^* = \frac{3}{2}nw^{*\,4}, \qquad \bar{u}_2^* = 12w^{*\,4}, \qquad \bar{u}_3^* = -24w^{*\,4}, \qquad \bar{u}_4^* = \frac{9}{2}w^{*\,4}.$$

The renormalization flow equations for the bare couplings of the generic RS theory were displayed in Ref. [17]. Using these equations, a new set of parameters g_i — the so-called nonlinear scaling fields introduced by Wegner [22] — can be defined with the following properties[8]:

- $g_i \equiv 0$ at the fixed point for all i.
- An infinitesimally small g_i, with all the others kept zero, gives an eigenvector belonging to the eigenvalue λ_i of the linearized renormalization group equations around the fixed point.
- They satisfy *exactly* the equations $\dot{g}_i = \lambda_i g_i$.

The RG flow of an observable y — the order parameter or an irreducible vertex, for instance — can be written in terms of the g_i's as follows:

$$\dot{y} = \left(k + \sum_i k_i g_i + \sum_{ij} k_{ij} g_i g_j + \cdots\right)y. \tag{44}$$

The solution of this equation, i.e. y in terms of the scaling fields is easily found:

$$y(g_1, g_2, \ldots) = g_1^{k/\lambda_1}\hat{y}\left(g_2 g_1^{-\lambda_2/\lambda_1}, \ldots, g_i g_1^{-\lambda_i/\lambda_1}, \ldots\right)$$
$$\times \exp\left(\sum_i \frac{k_i}{\lambda_i}g_i + \sum_{ij}\frac{k_{ij}}{\lambda_i+\lambda_j}g_i g_j + \cdots\right), \tag{45}$$

the scaling function $\hat{y}(\ldots)$ can be determined by perturbative methods.

In our two-parameter system defined by τ and w the two nonzero scaling fields[9] $r \equiv g_1$ and $\tilde{g} \equiv g_2$ (the notations are chosen to keep connection with previous sections) can be found by

---

[8] The summary presented in this paragraph about the use of nonlinear scaling fields is quite general, not limited to the nontrivial fixed point of the RS replica field theory.

[9] A second relevant scaling field g_3 with eigenvalue $\lambda_3 = 2 - \frac{2}{3}\epsilon + \cdots$, see [17], is also nonzero, though infinitesimal, arising from the small shift of the fluctuating fields to generate the nonzero order parameter RS system. The bare

starting with the RG equations (6)[10] and taking the temperature-like relevant eigenvalue λ_r and the irrelevant one, $\lambda_{\tilde{g}}$, from Ref. [17] as

$$\lambda_r \equiv \frac{1}{\nu} = 2 - \frac{5}{3}\epsilon + \cdots, \qquad \lambda_{\tilde{g}} = -\epsilon + \cdots. \tag{46}$$

The bare parameters are then straightforwardly expressed by the scaling fields as

$$w^2 = w^{*2} + \frac{\tilde{g}}{1 - 2\frac{\tilde{g}}{\epsilon}} = \frac{\epsilon/2}{1 - 2\frac{\tilde{g}}{\epsilon}},$$

$$\tau = r\left(1 - 2\frac{\tilde{g}}{\epsilon}\right)^{-5/3}. \tag{47}$$

5.2. x_1 below the upper critical dimension

For the calculation of x_1 to first order in ϵ, the RG study of q, w_1 and u_2 is required. The truncated (one-loop) renormalization group equations (9), (16) and (19) — see also footnote 10 — can be used whenever $w^2 \ll 1$ and $\tau \ll 1$. We can solve these truncated equations in a similar way as (45) was derived from the generic equation (44). The scaling exponents and the relations between bare and scaling parameters are taken from Eqs. (46) and (47), respectively. The scaling functions, which are always denoted by the "hat" symbol, cannot be determined by the RG equations alone, but the perturbative results of Eqs. (2), (4) and (5) make it possible to get them to first order in ϵ. [The bare values must be replaced by the scaling fields using (47), and take into account again footnote 10.] In the following, the results for q, w_1 and u_2 are listed in itemized form. The k and $k_2 \equiv k_{\tilde{g}}$ quantities defined in (44) are also presented for completeness.

-

$$q = r^{1+\frac{\epsilon}{2}}\hat{q}\left(\tilde{g}r^{\frac{\epsilon}{2}}\right) \times \left(1 - 2\frac{\tilde{g}}{\epsilon}\right)^{-1/6}, \tag{48}$$

with the scaling function

$$w^*\hat{q}(x) = \left[1 + \left(\frac{1}{2}\ln 2 + 1\right)\epsilon + \cdots\right] + 2\left[1 + \left(\ln 2 + \frac{17}{6}\right)\epsilon + \cdots\right]\left(\frac{x}{\epsilon}\right)$$
$$+ O\left[\left(\frac{x}{\epsilon}\right)^2\right]; \tag{49}$$

$$k = 2 - \frac{\epsilon}{2} + \frac{1}{2}\eta_L^* = 2 - \frac{2}{3}\epsilon + O(\epsilon^2) \quad \text{and} \quad k_{\tilde{g}} = -\frac{1}{3} + O(\epsilon). \tag{50}$$

---

parameters of this generic RS theory can be expressed in term of g_3 using the RG equations of Ref. [17]. The two most prominent examples are: $\bar{m}_2 = -g_3 + \cdots$ and $\bar{w}_2 = 12w^{*3}g_3 + \cdots$. As $g_3 \times g_1^{-\lambda_3/\lambda_1}$ is a RG invariant which is infinitesimal, g_3 itself remains infinitesimal in the regime of our approximation, justifying in this way the neglect of nonsymmetrical bare couplings from our RG equations. In references [7,20] an alternative scheme was used with a finite g_3 entering after appropriately redefining the field theory for getting rid of "tadpole" diagrams. (The irreducible vertices are the same in both approaches.)

[10] But be careful to replace $-|\epsilon|$ with ϵ.

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$$w_1 = r^{\frac{\epsilon}{2}} \hat{w}_1\left(\tilde{g} r^{\frac{\epsilon}{2}}\right) \times \left(1 - 2\frac{\tilde{g}}{\epsilon}\right)^{1/2}, \tag{51}$$

with the scaling function

$$\frac{\hat{w}_1(x)}{w^*} = \left[1 + \left(4\ln 2 - \frac{39}{4} - \frac{7}{2}\ln n\right)\epsilon + \cdots\right]$$
$$+ 2\left[1 + \left(8\ln 2 - \frac{56}{3} - 7\ln n\right)\epsilon + \cdots\right]\left(\frac{x}{\epsilon}\right) + O\left[\left(\frac{x}{\epsilon}\right)^2\right];$$
$$k = \frac{\epsilon}{2} - \frac{3}{2}\eta_R^* = \epsilon + O(\epsilon^2) \quad \text{and} \quad k_{\tilde{g}} = 1 + O(\epsilon). \tag{52}$$

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$$u_2 = r^{-1}\hat{u}_2\left(\tilde{g} r^{\frac{\epsilon}{2}}\right) \times \left(1 - 2\frac{\tilde{g}}{\epsilon}\right)^{2/3}, \tag{53}$$

with the scaling function

$$\frac{\hat{u}_2(x)}{w^{*4}} = 6\left[1 + O(\epsilon)\right] + 12\left[1 + O(\epsilon)\right]\left(\frac{x}{\epsilon}\right) + O\left[\left(\frac{x}{\epsilon}\right)^2\right];$$
$$k = -2 + \epsilon - 2\eta_R^* = -2 + \frac{5}{3}\epsilon + O(\epsilon^2) \quad \text{and} \quad k_{\tilde{g}} = \frac{4}{3} + O(\epsilon). \tag{54}$$

The following remarks are appropriate here: Firstly, according to Eqs. (45), (46) and (50) the temperature exponent for the order parameter q is *exactly* $k/\lambda_r = (2 - \epsilon/2 + \eta^*/2)\nu \equiv \beta = 1 + \epsilon/2 + O(\epsilon^2)$; see (48). The temperature exponents in (48), (51) and (53) are correct up to ϵ order. Secondly, the discussion below Eq. (32) concerning the fully replicon, infrared divergent contribution to w_1 is equally valid for the $\ln n$ terms in (52). Thirdly, the $O(\epsilon)$ corrections in the scaling functions $\hat{q}$ and $\hat{w}_1$ are unnecessary for the leading order calculation of x_1; they are displayed here to demonstrate the general form of the ϵ expansion. The corresponding corrections for $\hat{u}_2$ are not even available, see (54), as they would require a two-loop level calculation.

The leading contribution in the ϵ expansion for x_1 follows from substituting q, w_1 and u_2 from Eqs. (48), (49), and (51), (52), and (53), (54), respectively, into the basic formula in Eq. (42). A remarkably simple formula reflecting the invariance of x_1 under renormalization can be concluded:

$$x_1 = 6w^{*2}\hat{x}_1\left(\tilde{g} r^{\frac{\epsilon}{2}}\right) = 3\epsilon\hat{x}_1\left(\tilde{g} r^{\frac{\epsilon}{2}}\right), \quad \text{with the scaling function}$$
$$\hat{x}_1(x) = \left[1 + O(\epsilon)\right] + 2\left[1 + O(\epsilon)\right]\left(\frac{x}{\epsilon}\right) + O\left[\left(\frac{x}{\epsilon}\right)^2\right]. \tag{55}$$

5.3. Almeida–Thouless line for $d < 6$

The external magnetic field h^2 evolves under renormalization according to Eq. (26), with $|\epsilon|$ replaced by $-\epsilon$. The corresponding nonlinear scaling field $\tilde{h}^2$ has now the relevant eigenvalue

$$\lambda_{\tilde{h}^2} = 4 - \frac{\epsilon}{2} - \frac{\eta^*}{2} \equiv \frac{\delta\beta}{\nu}.$$

The flow equation for the replicon mass — Eq. (12) — does not contain explicitly the magnetic field, therefore it enters the solution only through the invariant $\tilde{h}^2 r^{-\delta\beta}$; see Eqs. (44), (45) and (46). According to the generic scheme (45), we have

$$\Gamma_R = r^{(2-\eta^*)\nu}\hat{\Gamma}_R\big(\tilde{g}r^{-\lambda_{\tilde{g}}\nu}, \tilde{h}^2 r^{-\delta\beta}, \dots\big) \times \exp\left(\frac{2}{3}\frac{\tilde{g}}{\lambda_{\tilde{g}}} + \cdots\right).$$

The exponential part can again be calculated in the truncated, one-loop approximation, in the usual way, providing (note that $\lambda_{\tilde{g}} = -\epsilon + \cdots$)

$$\left(1 - 2\frac{\tilde{g}}{\epsilon}\right)^{\frac{1}{3}},$$

whereas a comparison with the perturbative result (14) — after substituting the bare parameters by their corresponding nonlinear scaling fields [see Eq. (47) and also

$$h^2 = \tilde{h}^2\left(1 - 2\frac{\tilde{g}}{\epsilon}\right)^{1/6} \tag{56}$$

which follows from (26)] — gives the scaling function:

$$\hat{\Gamma}_R(x, y) = w^{*2}\bigg\{[-4 + O(\epsilon)] + [-24 + O(\epsilon)]\left(\frac{x}{\epsilon}\right) + [1 + O(\epsilon)]\left(\frac{y}{w^*}\right)$$
$$+ [-2 + O(\epsilon)]\left(\frac{x}{\epsilon}\right)\left(\frac{y}{w^*}\right) + \cdots\bigg\}. \tag{57}$$

The zero of the scaling function gives the AT line:

$$\tilde{h}^2 = 4w^* r^{\delta\beta} = 4w^* r^{2+\cdots}, \qquad \frac{\tilde{g}}{\epsilon}r^{\epsilon/2} \ll 1 \quad \text{and} \quad 0 < \epsilon \ll 1. \tag{58}$$

For the fixed point system, $w = w^*$ implies $\tilde{g} = 0$ and $\tilde{h}^2 = h^2$, $r = \tau$. The result in (58) is then identical with Eq. (18) of Ref. [7].

6. Discussion of the results and conclusions

For a thorough analysis of the d-dependence of x_1 while crossing the upper critical dimension, we recollect here the one-loop truncated results from previous sections; see Eqs. (22), (34) and (55). The goodness of these approximations depends on the smallness of the scaling variable, which is defined and expressed in terms of the bare parameters w^2 and τ as follows:

$$\text{scaling variable} = \begin{cases} \frac{2}{|\epsilon|}\tilde{w}^2 r^{|\epsilon|/2} = \frac{2}{|\epsilon|}w^2\tau^{|\epsilon|/2}(1 + \frac{2}{|\epsilon|}w^2)^{-1-\frac{5}{6}|\epsilon|}, \\ \quad d > 6, \text{ see } (8), \\ w^2(1 - w^2 \ln r)^{-1} = w^2(1 + \frac{5}{3}w^2 \ln w^2 - w^2 \ln \tau)^{-1}, \\ \quad d = 6, \text{ see } (30), \\ \frac{2}{\epsilon}\tilde{g}r^{\epsilon/2} = \tau^{w^{*2}}(\frac{w^{*2}}{w^2})^{\frac{5}{3}w^{*2}}(1 - \frac{w^{*2}}{w^2}), \qquad w^{*2} = \frac{\epsilon}{2}, \\ \quad d < 6, \text{ see } (47). \end{cases} \tag{59}$$

This scaling variable is displayed — for a chosen pair of bare values $w^2 = 0.005$ and $\tau = 0.0001$, both much smaller than one, as it should be in this approximation — as a function of dimension d
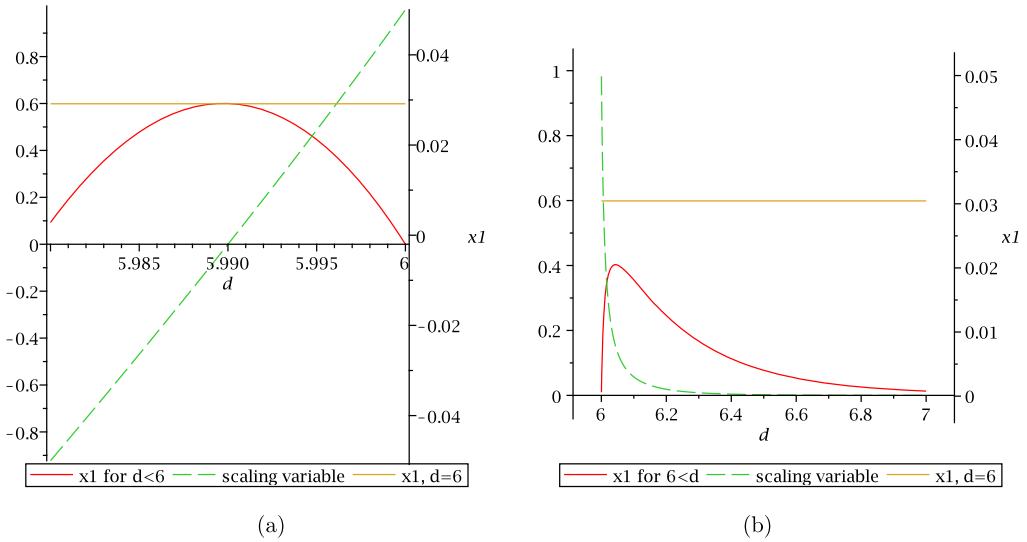
Fig. 2. The scaling variable (left vertical axis) measures the goodness of the approximation. (a): $d < 6$ and (b): $d > 6$. The dependence of x_1 is also shown in both regimes, together with its $d = 6$ value (horizontal lines). $w^2 = 0.005$ and $\tau = 0.0001$ are fixed in this figure. The approximation breaks down when approaching $d = 6$ from both sides.

below [Fig. 2(a)] and above [Fig. 2(b)] 6, where it takes ≈ 0.005. x_1 is also shown in this figure, with the awkward behaviour of approaching zero from both sides of the upper critical dimension six, while $x_1 \approx 0.03$ in $d = 6$. It is clear, however, from the figure that our approximation breaks down when approaching $d = 6$ from either side, as the scaling variable goes to unity in that limit. As a matter of fact, it must be stipulated that the scaling variable be at least as good as in $d = 6$, i.e. $\lesssim 0.005$. Therefore, the range of applicability of our approximation (for the given w and τ) is constrained to $d \approx 5.99$ and $d \gtrsim 6.4$, respectively. (Note that the chosen w is just the fixed point when $d = 5.99$.) Representative values of x_1 in these ranges, together with the six-dimensional case, are presented in Table 1. It can be concluded from this example that x_1 keeps on being monotonically increasing when lowering dimensions through 6. Nevertheless, a discontinuity of $x_1(w^2, \tau)$ at $d = 6$ cannot be excluded. An extrapolation of the data from the range $d \gtrsim 6.4$, using an exponential and/or a power law fit, provides $x_1(0.005, 0.0001) \approx 0.026$–$0.028$, a value somewhat lower than the six-dimensional one, 0.030, when considering the scaling variable as a measure of the relative error (it is ≈ 0.005 in six dimensions, see Table 1). A similar extrapolation from the $d < 6$ side, however, does not exist.

Below six dimensions x_1 has only a slight temperature dependence, and it becomes nonzero and universal at criticality:

$$x_1 = \left[3\epsilon + O\left(\epsilon^2\right)\right] + C\tau^{\frac{\epsilon}{2}+\cdots} + \cdots, \quad d = 6 - \epsilon;$$

C is a nonuniversal, i.e. w-dependent, amplitude. The typical behaviour for both below and above six dimensions is displayed in Fig. 3, the value of the cubic coupling is kept $w^2 = 0.005$. The tendency of an increasing x_1 while lowering the dimension is again obvious. The vertical scale was magnified in the left subfigure (a) to show the qualitative difference between the 6- and 5.99-dimensional curves.

The critical field along the AT line, for a given pair of bare parameters w^2 and τ, can be analysed using results from previous sections. See Eqs. (15), (28) for $d > 6$, and (37) for $d = 6$.

Table 1

x_1 around six dimensions shows monotonically increasing behaviour with decreasing dimensionality. The smallness of the scaling variable verifies the approximation. The bare parameters $w^2 = 0.005$ and $\tau = 0.0001$ are the same as in Fig. 2.

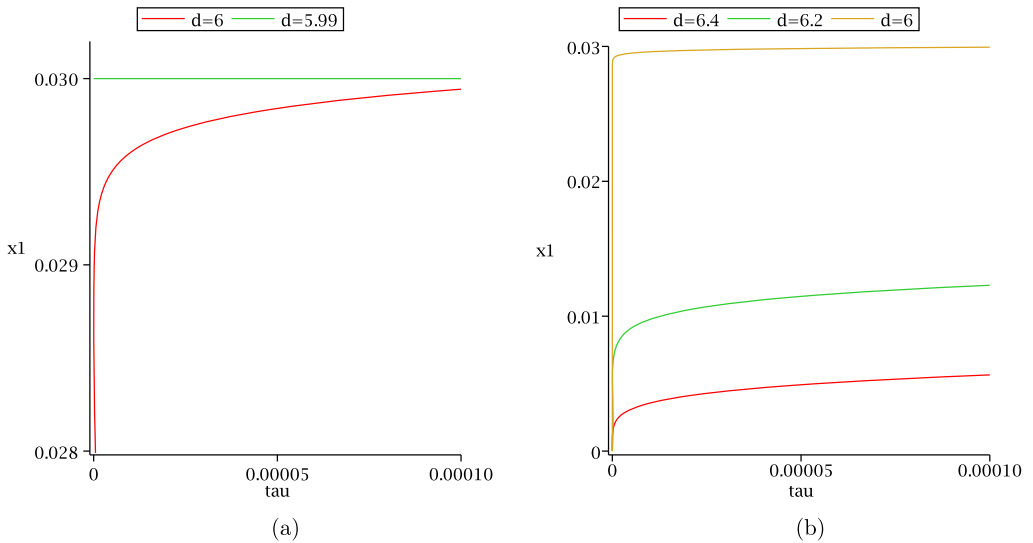| d	$x_1 \times 10^2$	Scaling variable
6.8	0.1287	0.000308
6.6	0.2648	0.001026
6.4	0.5649	0.003834
6	2.9943	0.004991
5.99005	2.9993	0.004776
5.99	3.0000	0
5.98995	3.0006	−0.004774

Fig. 3. x_1 as a function of the reduced temperature τ; $w^2 = 0.005$. (a): $d \leqslant 6$ and (b): $d \geqslant 6$. x_1 is zero at criticality, i.e. for $\tau = 0$, when $d \geqslant 6$. On the contrary, it is nonzero for $d < 6$ and has the universal value $x_1 = 3\epsilon + O(\epsilon^2)$ at T_c.

Below six dimensions, if we wish to move somewhat away from the fixed point, the zero of the expanded equation (57) must be found, providing [instead of (58)]:

$$\tilde{h}^2 = 4w^* r^{\delta\beta}\left(1 + 8\frac{\tilde{g}}{\epsilon}r^{\epsilon/2}\right), \qquad \frac{\tilde{g}}{\epsilon}r^{\epsilon/2} \ll 1 \quad \text{and} \quad 0 < \epsilon \ll 1.$$

Eqs. (47), (56) and (59), together with $\delta\beta = 2 + \frac{3}{2}\epsilon$, give us the critical field as

$$wh^2 = 4w^2\tau^{2+3w^{*2}}\left(\frac{w^{*2}}{w^2}\right)^{4+5w^{*2}}\left[1 + 4\tau^{w^{*2}}\left(\frac{w^{*2}}{w^2}\right)^{\frac{5}{3}w^{*2}}\left(1 - \frac{w^{*2}}{w^2}\right)\right],$$

$$w^{*2} = \frac{\epsilon}{2}.$$

The critical field where RSB sets in as a function of temperature (i.e. the AT line) — or more precisely wh^2 as a function of τ — is shown in Fig. 4 for three different dimensions at fixed cubic coupling $w^2 = 0.005$. The curve for $d = 5.99$ (note that the system is at exactly the fixed
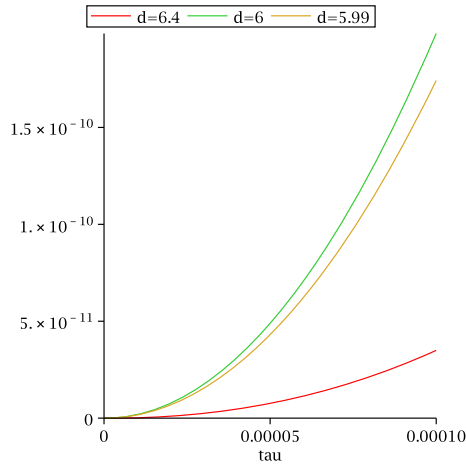
Fig. 4. Almeida–Thouless line (wh^2 versus τ) of the field theoretic model with $w^2 = 0.005$ for three different dimensions.

Table 2
Critical field values around six dimensions for $w^2 = 0.005$ and $\tau = 0.0001$. Below the critical field replica symmetry is broken. The scaling variable's values are, of course, the same as in Table 1.

d	$wh^2 \times 10^{10}$	Scaling variable
6.8	0.0827	0.000308
6.6	0.1680	0.001026
6.4	0.3497	0.003834
6	1.9849	0.004991
5.99005	1.7410	0.004776
5.99	1.7419	0
5.98995	1.7421	−0.004774

point then) is significantly below the six-dimensional one. It is easy to see that this follows directly from the exponent inequality $\delta\beta - 2 = \frac{3}{2}\epsilon + \cdots > 0$. To see clearly the behaviour of the critical field above and below six dimensions for decreasing d, it is tabulated in Table 2 for the system with $w^2 = 0.005$ and $\tau = 0.0001$. The last three rows of this table show that the kind of monotonicity found above six dimensions is restored below it, i.e. the critical field increases with decreasing dimensions. It must be remarked, however, that around the last dimension value $d = 5.98995$, the error[11] of our approximation starts to be the same order of magnitude as the variation of the critical field itself. The range where this one-loop perturbative method is applicable below the upper critical dimension is certainly very narrow.

As a conclusion, we can confidently claim that RSB survives below six dimensions in the cubic replica field theory representing the Ising spin glass. We focused on two quantities which are strongly related to RSB: the breakpoint of the order parameter function x_1 and the Almeida–Thouless line. A combination of the perturbative one-loop method with a simple two-parameter renormalization group (which is correct near the critical fixed point) provided reliable results in all the three ranges of dimensionalities, i.e. for d larger, equal, and less than six. The calculations

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[11] The relative error can be estimated as being proportional to the square of the scaling variable.

above and below six dimensions are rather different, due to the Gaussian versus nontrivial fixed point governing critical behaviour in the two cases. The applied perturbative method makes it impossible to approach closely the upper critical dimension: the range of dimensions where the approximation is correct for a given system (i.e. for given w and τ) is very narrow and close to 6 when $d < 6$, whereas it is $d \gtrsim 6.2$ when $d > 6$ (and the farther we are from $d = 6$, the better the approximation). The six-dimensional case needs special care along the way systems at their upper critical dimension are commonly studied [18]. The logarithmic temperature dependences obtained are quite similar to those in ordinary systems at their upper critical dimension.

Above six dimensions both x_1 and the critical field are monotonically increasing for decreasing d, and this tendency persists for $d < 6$. There seems to be, however, a discontinuity of the critical field at $d = 6^-$: the AT line for $d \lesssim 6$ is significantly below the six-dimensional one, see Fig. 4 and Table 2. Nevertheless, we can notice that the trend of increasing dominance of RSB for decreasing space dimensions persists even below six dimensions.

As a final remark, we recall that for $d < 6$, x_1 gains the qualitatively new feature of being nonzero (and universal!) at criticality. This might suggest a kind of first order transition. That this is not the case can be clearly seen by displaying the order parameter function using Eqs. (41) and (43):

$$q(x) = q_1 \hat{q}(x/x_1), \quad \text{where } q_1 \sim q \sim \tau^\beta.$$

An elaboration of the equation of state along the lines of Section 4 for $d < 6$ (which is out of the scope of the present paper, and is left for a future publication), proves that, next to the spin glass transition, $\hat{q}$ is a function independent of temperature,[12] and thus nontrivial even at criticality. The prefactor q_1, however, disappears at T_c ensuring continuity of the order parameter through the spin glass transition.

Acknowledgement

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